Arakelov-type inequalities
for Hodge bundles

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§ 0.   Introduction

The inequalities from the title refer back to Arakelov's article [Arakelov]. The main result of that paper is:

Theorem. *Fix a complete curve C of genus > 1 and a finite set S of points on C. There are at most finitely many non-isotrivial families of curves of given genus over C that are smooth over $C \setminus S$.*

The proof consists of two parts. First one proves that there are only finitely many such families (this is a *boundedness* statement) by bounding the degree d of the relative canonical bundle in terms of the genus p of C, the genus g of the fiber and the cardinality of the set S :

$$ 0 \le d \le (2p - 2 + \#S)\frac{g}{2}. $$

The second part consists of establishing *rigidity* for a non-isotrivial family. It follows upon identifying the deformation space of the family with the H^1 of the inverse of the relative canonical bundle, which is shown to be ample. Kodaira vanishing then completes the proof.

This approach can be carried out for other situations as well. In fact [Faltings] deals with the case of abelian varieties and shows that boundedness always holds and that for rigidity one has to impose further conditions besides non-isotriviality. Subsequently the rigidity statement has been generalized in [Peters90] and using his result, the case of K3-surfaces, resp. Abelian varieties could be treated completely by Saito and Zucker in [Saito-Zucker], resp. by Saito in [Saito].

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The boundedness statement is an inequality for the degree d of the direct image of the relative canonical bundle, *i.e.*, the (canonical extension) of the Hodge bundle and one can ask for bounds for the degrees of the other Hodge bundles. In fact, the main result of this note gives such a bound for the Hodge components of complex variations of Hodge structures in terms of ranks of iterates of the Higgs field (the linear map between Hodge components induced by the Gauss-Manin connection). By way of an example, we have :

Proposition. *Let* $V = \bigoplus_{p=0}^{w} V^{p,w-p}$ *be a real weight w variation of Hodge structures over a punctured curve $C \setminus S$ with unipotent local monodromy-operators. Let* $\sigma^p : V^{p,w-p} \to V^{p-1,w-p+1}$ *the k-th component of the Higgs field and* $\sigma^k = \sigma_{w-k+1} \circ \cdots \circ \sigma_w : V^{w,0} \to V^{w-k,k}$ *the k-th iterate. Then*

$$0 \le \deg V^{w,0} \le (2p - 2 + \#S) \left(\sum_{r=1}^{w} \frac{r}{2} \left(\operatorname{rank} \sigma^{r-1} - \operatorname{rank} \sigma^r \right) \right).$$

Some time ago I sketched a proof of a similar, but weaker inequality in [Peters86], but the details of this proof never appeared. Deligne found an amplification of my argument when the base is a *compact* curve leading to optimal bounds for *complex* variations of Hodge structures (letter to the author 18/2/1986). The principal goal of this note is to give a complete proof of the refined inequalities (for a complex variation with quasi-unipotent local monodromy-operators over a quasi-projective smooth curve) based on this letter in the light of later developments which I sketch below.

The reason for writing up this note stems from a recent revival of interest in this circle of ideas: on the one hand Parshin posed me some questions related to this. On the other hand, Jost and Zuo sent me a preprint [JostZuo] containing similar (but weaker) bounds obtained by essentially the same method.

Continuing with the historical development, Deligne's letter and subsequent correspondence between Deligne and Beilinson together with basic ideas and results of Hitchin paved the way for the theory of Higgs bundles, the proper framework for such questions. See [Simpson92], [Simpson94] and [Simpson95] for a further discussion of these matters.

In relation with Simpson's work, I should remark that the boundedness result from [Simpson94] (Corollary 3.4 with $\mu = 0$ and $P = 0$) immediately implies that the Chern numbers of Hodge bundles underlying a complex variation of given type over a compact projective variety can only assume finitely many values. It follows that there are *a priori* bounds on these Chern numbers. If the base is a (not necessarily compact) curve, Simpson shows in [Simpson90] that the same methods give boundedness for the canonical extensions of the Hodge bundles. Explicit bounds were not given however. Conversely, knowing that bounds on the degrees of the Hodge bundles exist can be used to simplify the rather technical proof for boundedness, as outlined in §4.

As to the further contents of this note, in §1 I rephrase known bounds on the curvature of Hodge bundles (with respect to the Hodge metric) in a way that shows how to adapt these in the case of non-compact algebraic base manifolds. The main argument is given in §2 while in §3 I explain what one has to change in the non-compact case.

In closing, I want to mention Eyssidieux work which treats a generalization of the true Arakelov inequality (weight one Hodge structures on a compact curve) to higher dimensional base manifolds. The inequalities concern variations of Hodge structure over a compact Kähler manifold M such that the period map is generically finite onto its image. See [Eyss].

§ 1. Curvature bounds for Hodge bundles

Let me recall that a *complex variation of Hodge bundles of weight w* on a complex manifold M consists of a complex local system V on M with a direct sum decomposition into complex subbundles

$$V = \bigoplus_{p+q=w} V^{p,q}$$

with the property that the canonical flat connection ∇ satisfies the transversality condition

$$(*) \qquad \nabla : V^{p,q} \to \mathcal{A}^{1,0}(V^{p-1,q+1}) \oplus \mathcal{A}^1(V^{p,q}) \oplus \mathcal{A}^{0,1}(V^{p+1.q-1}).$$

The local system V defines a holomorphic vector bundle denoted by the same symbol. The bundles $V^{p,q}$ are not necessarily holomorphic, but the transversality condition implies that the filtration defined by

$$F^p V := \bigoplus_{r \geq p} V^{r,w-r}$$

is holomorphic. We have a C^∞-isomorphism between the *Hodge bundle*

$$V_{\text{Hdg}} := \bigoplus_p F^{p+1}V / F^p V$$

and the bundle V which we shall use to transport the connection to the former.

Of course, the usual (real) variations are examples of complex variations. These satisfy the additional *reality constraint* $V^{p,q} = \bar{V}^{q,p}$. Conversely, a complex variation V together with its complex conjugate, defines a real variation of Hodge structures on $V \oplus \bar{V}$ in the obvious manner. In passing, we observe however that we can have complex variations of any given pure type. If for instance $V^{p,q}$ has the property that it is preserved by ∇ *i.e.*, if it is a flat subbundle, it is itself a complex variation.

One says that V is *polarized* by a bilinear form b if b is preserved by ∇ and the two Riemann bilinear equations are verified:

$$b(u,v) = 0 \quad u \in H^{p,q}, v \in H^{r,s}, \, (u,v) \neq (s,r)$$
$$h_C(u,v) = (\text{i})^w b(Cu, \bar{v}) \text{ is a positive definite hermitian metric.}$$

Here C is the Weil-operator which equals multiplication with i^{p-q} on $H^{p,q}$. Instead of b one can also consider the hermitian form

$$h(u,v) = \bigoplus_p (-1)^p h_C(u,v) = \text{i}^w b(u, \bar{w}),$$

preserved by ∇. As before, we shall transport b, h and h_C to V_{Hdg} using the C^∞-isomorphism $V \xrightarrow{\sim} V_{\text{Hdg}}$.

If we decompose

$$\nabla = \sigma^- + D + \sigma^+$$

according to the transversality condition (*), the operator D is a connection on V_{Hdg} and the operators σ^+ and σ^- are $\mathcal{A}_M^0$-linear. Moreover, the operator σ^+ is the h_C-conjugate of σ^- so that one may write

$$\sigma := \sigma^-$$
$$\sigma^* = \sigma^+$$

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The $\overline{\partial}$-operator for the holomorphic structure on V_{Hdg} is given by the $(0,1)$-part $D^{0,1}$ of D and D preserves the metric h_C. So D is the *Chern connection*, *i.e.*, the unique metric connection on the Hodge bundle whose $(0,1)$-part is the $\overline{\partial}$-operator.

Decomposing the equation $\nabla \circ \nabla = 0$ into types yields various equalities. The first

$$0 = \overline{\partial}(\sigma) := \overline{\partial} \circ \sigma + \sigma \circ \overline{\partial}$$

says that σ is a *holomorphic* endomorphism. The second

$$\sigma \circ \sigma = 0$$

implies that the pair $(V_{\mathrm{Hdg}}, \sigma)$ is a so-called *Higgs bundle*. By definition this is a pair (E, θ) consisting of a holomorphic bundle E with a holomorphic map $\theta : E \to \Omega^1(E)$ satisfying $\theta \wedge \theta = 0$ in $\Omega^2 \operatorname{End}(E)$. The endomorphism θ is also called the *Higgs field*. Recall also that a hermitian metric k on a Higgs bundle is called *harmonic* if its Chern connection D_k combines with θ and its k-conjugate θ^* to give a flat connection $\theta + D_k + \theta^*$. In our case, the Hodge metric h_C is indeed harmonic.

Summarizing the preceding discussion, from a polarized complex variation of Hodge bundles over M we have constructed a Higgs bundle equipped with a harmonic metric. If M is compact, such bundles can be shown to be semi-stable. Let me recall that a harmonic Higgs bundle (V, h) is *stable* resp. *semi-stable* if for any *proper* Higgs *subsheaf* $W \subset V$ *i.e.*, a coherent submodule preserved by the Higgs field, one has an inequality of slopes

$$\mu(W) < \mu(V), \text{ resp. } \mu(W) \le \mu(V).$$

The *slope* for vector bundle E on a Kähler manifold (M, ω) is

$$\mu(E) = \deg(E) / \operatorname{rank}(E),$$

with the degree of E is defined using the Kähler metric:

$$\deg(E) = c_1(E) \cdot [\omega]^{m-1}, \qquad m = \dim M.$$

Note that $\deg(E) = \deg(\bigwedge^r E)$, $r = \operatorname{rank} E$. For a torsion free coherent sheaf E, one has to replace E by the line bundle which is the double dual of $\bigwedge^r E$.

Semi-stability of harmonic Higgs bundles can be proved as a consequence of the curvature formula, which is a straightforward consequence of the transversality relation:

1.1. Lemma. *The curvature of the Chern connection D on V_{Hodge} is given by:*

$$F_D = -[\sigma, \sigma^*].$$

Below (see Lemma 1.4) I give a proof of a refined version of the semi-stability property. Let me observe however that semi-stability is part of the complete characterization of harmonic Higgs bundles as found by Simpson (see [Simpson92]):

Theorem. *A Higgs bundle over a* **compact** *Kähler manifold M with a harmonic metric is the direct sum of stable Higgs bundles with the same slope. Any local system which is the direct sum of irreducible local systems admits the structure of a Higgs bundle with a harmonic metric. The category of Higgs bundles admitting a harmonic metric is equivalent to the category of semi-simple local systems, i.e., those that are direct sums of irreducible local systems.*

Observe that a Higgs bundle has zero Chern classes since it carries a flat connection and hence semi-stability means that any Higgs subbundle has non-positive first Chern class. I need a refinement of this in terms of the *first Chern forms*

$$\gamma_1(E, h) = \frac{\mathrm{i}}{2\pi} \cdot (\text{Trace of the curvature of the Chern connection}).$$

So I need to estimate the curvature (with respect to the Hodge metric) of a *graded Higgs subbundle* $W = \bigoplus_p W^{p,w-p}$, i.e., $W^{p,w-p} \subset V^{p,w-p}$ and $\sigma W^{p,w-p} \subset \Omega_M^1(W^{p-1,w+p+1})$. The curvature estimate we are after reads:

1.2. Lemma. *For a subsystem $W \subset V$ we have*

$$\mathrm{i}\,\mathrm{Tr}[\sigma|W_{\mathrm{Hdg}}, (\sigma|W_{\mathrm{Hdg}})^*] \geq 0$$

with equality everywhere if and only if the orthogonal complement $W_{\mathrm{Hdg}}^\perp$ with respect to h_C is preserved by σ so that we have a direct sum decomposition of Higgs bundles

$$W_{\mathrm{Hdg}} \oplus W_{\mathrm{Hdg}}^\perp,$$

or, equivalently, of complex systems of Hodge bundles.

Proof: If we split $V = W \oplus W^\perp$ into a C^∞-orthogonal sum with respect to the Hodge metric and write σ into block-form, the fact that σ preserves W means that this block-form takes the shape

$$\sigma = \begin{pmatrix} S & T \\ 0 & S' \end{pmatrix}.$$

Then $\mathrm{Tr}[\sigma|W, (\sigma|W)^*] = \mathrm{Tr}(S \circ S^* - S \circ S^* + T \circ T^*) = \mathrm{Tr}\,T \circ T^*$. If we multiply this with i this is a positive definite $(1,1)$-form. The trace is zero if and only if $T = 0$ which means that σ preserves also $W^\perp$.

The last clause is a consequence of the following discussion.   Q.E.D.

Let me compare the flat structure (V, ∇) and the holomorphic structure $(V_{\mathrm{Hdg}}, \sigma)$ using the *principle of pluri-subharmonicity* : on a compact complex manifold there are only constant pluri-subharmonic functions. The result is:

1.3. Lemma. *Suppose M is a smooth projective variety. A holomorphic section of V_{Hdg} satisfies $\sigma(s) = 0$ if and only if $\nabla(s) = 0$. In other words the flat sections are precisely the holomorphic sections of the Hodge bundle killed by the Higgs field.*

Proof: The well known Bochner-type formula (see [Schmid, §7])

$$\begin{aligned} \partial\bar{\partial}h_C(s,s) &= h_C(D_h s, D_h s) - h_C(F_h s, s) \geq h_C([\sigma(s), \sigma^*(s)], s) \\ &= -h_C(\sigma(s), \sigma(s)) + h_C(\sigma^*(s), \sigma^*(s)) \end{aligned}$$

shows that when $\sigma(s) = 0$, the function $h_C(s, s)$ is pluri-subharmonic and so constant. Hence $D_h s = 0 = \sigma^*(s)$, implying $\nabla(s) = 0$. Conversely, a flat holomorphic section satisfies $\sigma(s) = 0$ by type considerations.   Q.E.D.

1.4. Corollary. *A complex subbundle $W \subset V$ is a subsystem, i.e., $\nabla W \subset W \otimes \Omega_X^1$ if and only if $W_{\mathrm{Hdg}} \subset V_{\mathrm{Hdg}}$ is a Higgs subbundle.*

Noticing the minus sign in the curvature of the Hodge metric and recalling that the curvature decreases on subbundles with equality if and only if the quotient bundle is holomorphic, Lemma 1.2 implies:

1.5. Corollary. *The first Chern form of a graded Higgs subbundle W of V (with respect to the Hodge metric) is negative semi-definite and it is zero everywhere if and only if $W^\perp$ is a (holomorphic) graded Higgs subbundle as well. In this case the complex variation splits as complex polarized variations of Hodge structures $V = W \oplus W^\perp$.*

1.6. Remark. These point-wise estimates can be integrated over any compact Kähler variety M showing that the slope of a Higgs subbundle is non-positive thereby proving semi-stability.

One also needs to know what happens for a morphism $f : V_1 \to V_2$ of graded Higgs bundles. In general neither the kernel nor the image are bundles, although they are preserved by the Higgs fields. Since the kernel of f is torsion free, its degree is well-defined. For the image this is not necessarily the case. We have to replace it by its saturation. Recall that a subsheaf $F \subset V$ of a locally free (or torsion free) sheaf V is saturated if V/F is torsion free. Any subsheaf $F \subset V$ is a subsheaf of a unique saturated subsheaf F^{sat}, its *saturation* which is defined as the inverse image under $V \to V/F$ of the torsion of the target. For Higgs bundles on *curves* the natural morphisms

$$\sigma_p : V^{p,w-p} \to V^{p-1,w+1-p} \otimes \Omega_M^1$$

then can be used to define image bundles

$$\mathrm{Im}(\sigma_p) = \left(\sigma_p V^{p,w-p}\right)^{\mathrm{sat}} \otimes (\Omega_M^1)^\vee.$$

§ 2.   A bound for variations over a compact curve

In this section I investigate polarized complex systems of Hodge bundles over a compact curve. The bundles $V^{p,q}$ can be given holomorphic structures via the C^∞-isomorphism $V^{p,q} \xrightarrow{\sim} F^p/F^{p+1} \subset V_{\mathrm{Hdg}}$ and in the sequel we tacitly make this identification.

2.1. Theorem. *Let $V = \bigoplus_p V^{p,w-p}$ be a weight w polarized complex system of Hodge over a compact curve C of genus p. Put*

$$\chi(C) = -\deg\left(\Omega_C^1\right) = 2 - 2p.$$

Let $\sigma = \bigoplus \sigma_p$ be the associated Higgs field and let $\sigma^k : V^{p,w-p} \to V^{p-k,w-p+k}$ the composition $\sigma_{p-k+1} \circ \cdots \circ \sigma_p$. Then we have the inequality

$$\deg V^{p,w-p} \leq -\chi(C) \sum_{r \geq 1} \frac{r}{2} \left(\mathrm{rank}\, \sigma^{r-1} - \mathrm{rank}\, \sigma^r\right)$$

Equality holds if and only if for some $k \geq 0$ the maps $\sigma_p, \ldots, \sigma_{p-k}$ are all isomorphisms and σ_{p-k-1} is zero. In this case $V^{p,w-p} \oplus \cdots \oplus V^{p-k-1,w-p+k+1}$ is a complex subvariation.

Proof (Deligne): Consider the following graded Higgs subbundles $W_r \subset V$, $r = 1, 2, \ldots$ which only are non-zero in degrees $p, \ldots, p - r + 1$ and which — using the notation of the image under σ as defined in the previous section — are given by

$$W_r^{p,w-p} := \operatorname{Ker} \sigma^r \subset V^{p,w-p}, \quad W_r^{p-1,w-p+1} := \operatorname{Im}(\sigma \operatorname{Ker} \sigma^r) \subset V^{p-1,w-p+1}, \ldots$$
$$\ldots \quad W_r^{p-r+1,w-p+r-1} := \operatorname{Im}(\sigma^{r-1} \operatorname{Ker} \sigma^r) \subset V^{p-r+1,w-p+r-1}.$$

Apply the stability property to these bundles. For simplicity we put

$$d_r = \deg\left(\operatorname{Ker} \sigma^r / \operatorname{Ker} \sigma^{r-1}\right)$$
$$l_r = \dim\left(\operatorname{Ker} \sigma^r / \operatorname{Ker} \sigma^{r-1}\right) = \operatorname{rank} \sigma^{r-1} - \operatorname{rank} \sigma^r$$

Now compute the degree of each of these r bundles $W_r^{p-k,w-p+k}$ using the isomorphism

$$\operatorname{Ker} \sigma^r / \operatorname{Ker} \sigma^k \xrightarrow{\sim} \sigma^k(\operatorname{Ker} \sigma^r)^{\operatorname{sat}} \subset V^{p-k,w-p+k} \otimes \left(\Omega_C^1\right)^{\otimes k}.$$

The bundle on the left has rank $l_r + \cdots + l_{k+1}$ and degree $d_r + \cdots + d_{k+1}$. It follows that $\deg W_r^{p-k,w-p+k} = \deg\left(\operatorname{Im}(\sigma^k \operatorname{Ker} \sigma^r)\right) = \deg\left((\operatorname{Ker} \sigma^r / \operatorname{Ker} \sigma^k) \otimes \left(\Omega_C^1\right)^{-\otimes k}\right) = d_r + \cdots + d_{k+1} + (l_r + \cdots + l_{k+1})k\chi$ and adding these for $k = 0, \ldots, r - 1$ one finds the total degree

$$\deg W_r = \left(\sum_{p=1}^{r} p d_p + l_p \frac{p(p-1)}{2} \cdot \chi(C)\right) \leq 0.$$

Now take a weighted sum of these inequalities in order to make appear $\deg V^p = \sum d_p$. Indeed

$$0 \geq \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \cdot \left(\sum_{p=1}^{k} p d_p + l_p \frac{p(p-1)}{2} \cdot \chi(C)\right) = \sum_{p \geq 1} d_p + \sum_{p \geq 1} l_p \frac{p-1}{2} \cdot \chi(C)$$

You have equality if and only if all of the equalities you started out with are equalities which means that the W_r are direct factors as graded Higgs subbundle of V. This translates into the σ_r being either zero or an isomorphism. To show this, consider the bundle W_1, *i.e.*, $\operatorname{Ker} \sigma_p \subset V^{p,w-p}$. If this is direct factor of V as a graded Higgs bundle, the orthogonal complement has the structure of a graded Higgs bundle. Since the latter, if not zero, must have type different from $(p, w - p)$ it follows that σ_p is either 0 or has maximal rank. If it is the zero map, the kernel $V^{p,w-p}$ is itself a subvariation. If it has maximal rank, its kernel is zero and hence has degree 0 and we repeat the argument with σ_{p-1}. If this map is zero $V^{p,w-p} \oplus V^{p-1,w-p+1}$ is a subvariation and if σ_{p-1} is an isomorphism we continue the procedure. In this way we produce a chain $\sigma_p, \ldots, \sigma_{p-k}$ of isomorphisms such that the next map σ_{p-k-1} is zero as asserted. Q.E.D.

Next, we remark that these bounds applied to the *dual* variation of Hodge structure gives bounds in the other direction. We shall make these explicit for real variations, which are self-dual in the obvious sense. In fact, the reality constraint $H^{p,q} = \bar{H}^{q,p}$ implies $\deg H^{p,q} = -\deg H^{q,p}$. In the real case, again one has equality only for a consecutive string $\sigma_p, \ldots, \sigma_{p-k}$ of isomorphisms such that the next one is zero. Note however that the complex subvariation splitting the real variation itself need not be real. The reality constraint itself imposes however some additional constraints.

2.2. Corollary. *Let V be a variation of polarized real weight w Hodge structures over a compact curve C with Higgs field $\sigma = \bigoplus \sigma_p$. Put $\bar{\sigma}_p = \sigma_{w-p}$ and $\sigma^k = \sigma_{p-k-1} \circ \cdots \circ \sigma_p : V^{p,w-p} \to V^{p-k,w-p+k}$ and $\bar{\sigma}^k = \bar{\sigma}_{p-k-1} \circ \cdots \circ \bar{\sigma}_p : V^{w-p,p} \to V^{w-p+k,p-k}$. Then we have the bounds*

$$\chi(C) \sum_{r \geq 1} \frac{r}{2} \left(\operatorname{rank} \bar{\sigma}^{r-1} - \operatorname{rank} \bar{\sigma}^r \right) \leq \deg V^{p,w-p} \leq$$

$$- \chi(C) \sum_{r \geq 1} \frac{r}{2} \left(\operatorname{rank} \sigma^{r-1} - \operatorname{rank} \sigma^r \right).$$

In particular, if $V^{p,w-p} \neq 0$ precisely in the interval $p = 0, \ldots, w$, the Hodge bundle $V^{w,0}$ satisfies the bounds

$$0 \leq \deg V^{w,0} \leq -\chi(C) \sum_{r=1}^{w} \frac{r}{2} \left(\operatorname{rank} \sigma^{r-1} - \operatorname{rank} \sigma^r \right)$$

and equality on the left holds if and only if $V^{w,0}$ is a flat subbundle, while equality on the right holds if and only if all the maps σ_k are isomorphisms.

Let me translate this Corollary in terms of period maps. The period domains involved are the Griffiths domain D resp. D' parametrizing real polarized Hodge structures of weight w of fixed given type, resp. the partial Hodge flags $V^{w,0} \subset V$ of the same type. Giving a polarized variation of Hodge structures of this type over the curve C is the same as giving its period map

$$p : C \to D/\Gamma,$$

where Γ is the monodromy group. Let me refer to [Griffiths] for the necessary background. Likewise, the partial Hodge flag is given by the partial period map

$$p' : C \to D'/\Gamma'$$

defined as the composition of p and the forgetful map $q : D/\Gamma \to D'/\Gamma'$. The subbundle $V^{w,0} = F^w$ being flat means exactly that p' is constant. At the other end of the spectrum, using that the maps σ_k measure the derivative of the period map, the latter is everywhere an immersion if and only if all σ_k are isomorphisms. Summarizing:

2.3. Corollary.

i) *One has $\deg W^{w,0} = 0$ if and only if the image of the full period map lands in one of the fibers of q.*

ii) *The equality $\deg V^{w,0} = -\chi(C) \sum_{r=1}^{w} \frac{r}{2} \left(\operatorname{rank} \sigma^{r-1} - \operatorname{rank} \sigma^r \right)$ holds if and only if the perod map is everywhere an immersion.*

2.4. Remark. Put

$$v^{p,q} := \dim V^{p,q}$$

$$v_0^{p,q} := \dim \operatorname{Ker}(\sigma_p : V^{p,q} \to V^{p-1,q+1}).$$

Using the estimate

$$\operatorname{rank} \sigma_{w-r} \leq \operatorname{rank} \sigma^{r+1}$$

and a telescoping argument, the above inequality implies the (a priori weaker) bound

$$\deg E^{w,0} \leq -\chi(C) \cdot \sum_{p} (v^{p,w-p} - v_o^{p,w-p})$$

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and this is due to [JostZuo] (they combine this with the obvious symmetry among the Hodge numbers and, similarly, the numbers $v_0^{p,q}$).

§ 3.   The case of a smooth non-compact curve

We let C be a smooth compactification and we let $S = C \setminus C_0$. Assume now that V_0 is a complex variation of Hodge structure with the property that the local monodromy operators γ_s around the points $s \in S$ are quasi-unipotent. This is true if the complex variation is defined over the integers, e.g. if the variation comes from geometry. This is the monodromy theorem, whose proof can be found in [Schmid].

The assumption imply that V_0 admits a vector bundle extension V over C and all the Hodge bundles $F^p V_0$ extend to holomorphic bundles $F^p V \subset V$ on C. The Higgs fields are now maps

$$\sigma : V \to V \otimes \Omega^1_C(\log S).$$

Asymptotic analysis of the Hodge metric h_p on every Hodge bundle F^p around the punctures shows that its Chern form is integrable and that there are non-negative rational numbers α_s^p, the *residues* of the operators γ_p such that

$$\deg\left(F^p\right) = \int_C \gamma_1(F^p, h_p) + \sum_{s \in S} \alpha_s^p.$$

For details of this I refer to [Peters84]. The residues are always zero in the unipotent case and conversely, if the Hodge bundles are *flat* the vanishing of all the residues implies that the local monodromy operators are all unipotent. In general, the residue is given by

$$(*) \qquad\qquad \alpha_s^p = \sum_{\alpha \in [0,1)} \alpha \dim\left(F^p \cap V_\alpha\right)$$

where V_α is the subspace of V on which the local monodromy operator T_s acts with eigenvalue $\exp(2\pi i \alpha)$.

The estimates of the previous section go through for the first summand and also for its successive quotients, the Hodge bundles. So we find

3.1. Theorem. *Let V be a complex variation over a curve $C_0 = C \setminus S$ with quasi-unipotent local monodromy operators. Put* genus$(C) = g$. *Let $\sigma = \bigoplus \sigma_p$ be the associated Higgs field and let $\sigma^k : V^{p,w-p} \to V^{p-k,w-p+k}$ the composition $\sigma_{p-k+1} \circ \cdots \circ \sigma_p$. Then the degree of the Hodge bundles are bounded by*

$$\deg V^{p,w-p} \leq (2g - 2 + \#S) \cdot \sum_r \frac{r}{2}\left(\operatorname{rank}\sigma^{r-1} - \operatorname{rank}\sigma^r\right) + \sum_{s \in S}(\alpha_s^p - \alpha_s^{p+1}),$$

with α_s given by (*).

If the local monodromy-operators are unipotent, the last summand vanishes and one has a description for those variations where the bound is attained:

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3.2. Addition. *In the situation of the preceding theorem, assume that the local monodromy operators are unipotent. Equality is attained if and only if for some $k \geq 0$ the maps $\sigma_p, \ldots, \sigma_{p-k}$ are all isomorphisms and σ_{p-k-1} is zero. In this case $V_0^{p,w-p} \oplus \cdots \oplus V_0^{p-k-1,w-p+k+1}$ is a complex subvariation of V_0.*

The proof is the same as before; it depends on the validity of Lemma 1.3 in the case of non-compact curves. The principle of pluri-subharmonicity needed here can be found in [Schmid], proof of Theorem (7.22). It really depends on his SL(2)-orbit theorem and so it is considerably less trivial.

3.3. Remark. For real variations there are bounds as in Corollary 2.2:

$$0 \leq \sum_{s \in S} \alpha_s^p \leq \deg V^{w,0} \leq (2p - 2 + \#S) \cdot \sum_{r=1}^{w} \frac{r}{2} \left(\operatorname{rank} \sigma^{r-1} - \operatorname{rank} \sigma^r \right) + \sum_{s \in S} \alpha_s^p$$

and if for instance $\deg V^{w,0} = 0$, all the local monodromy operators are unipotent and $V^{w,0}$ is a flat subbundle of V. See also [Peters84].

§ 4. Boundedness revisited

Here we give the proof that a bound on the degrees of all of the Hodge bundles imply boundedness. To be precise, we need both upper and lowerbounds so that only finitely many degrees are possible. We repeat here that lower bounds are obtained from the upper bounds of the dual variation.

First we need to see that only those variations that occur on a fixed local systems need to be considered. For this we invoke Deligne's boundedness result from [Deligne]:

Theorem. *Given a smooth algebraic variety S there are only finitely many classes of representations of $\pi_1(S)$ on a given rational vector space V such that the resulting flat vector bundle underlies a polarizable complex variation of Hodge structure of a given weight.*

I would like to point out that the proof of this theorem is not too difficult, except for one point: it uses the (by now standard fact) that a polarizable complex variation of Hodge structure is semi-simple; in any case it is much easier than Simpson's methods used to show boundedness.

We complete the argument by showing:

4.1. Lemma. *Let there be given a flat vector bundle on a curve C and a finite set of integers $\{d_p\}$, $p \in I$. Complex variations $V = \bigoplus_{p \in I} V^{p,w-p}$ with $\deg V^{p,w-p} = d_p$ are parametrized by an open set of an algebraic variety.*

Proof: Consider the bundle $\mathcal{F}$ over C whose fiber over c consists of all flags in V_c, the fiber of the given complex variation V at c, which are of the same type as the Hodge flags. Then the total space $\mathcal{F}$ is a projective variety. There is a subvariety $\mathcal{G}$ of $\mathcal{F}$ formed by flags satisfying the first bilinear relation. Over $\mathcal{F}$ we have the tautological flag of vectorbundles F^p. A complex variation underlying V is the same thing as a section s of $\mathcal{G}$ which in each fiber lands in the set of flags satisfying the second period relation. The degree of the p-th Hodge bundle is the degree of $s^* F^p$,

together making up the *flag degree* $\{d_p\}$, $p \in I$. Since these are all fixed, the desired collection of complex variations forms an open set in the variety S of sections of the partial flag bundle $\mathcal{G}$ over C of fixed flag degree. This variety S can be viewed as the Hilbert scheme of curves in $\mathcal{G}$ of fixed multidegree with respect to the embeddings defined by the flags.   Q.E.D.

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